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Evaluating Fractional Derivatives of Two Matrix Fractional Functions Based on Jumarie Type of Riemann-Liouville Fractional Derivative

Chii-Huei Yu

School of Mathematics and Statistics, Zhaoqing University, Guangdong, China

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Abstract: **In this paper, based on Jumarie's modified Riemann-Liouville (R-L) fractional derivative and a new multiplication of fractional analytic functions, we obtain arbitrary order fractional derivative of two matrix fractional functions. Fractional binomial theorem plays an important role in this article. In fact, our results are generalizations of ordinary calculus results.**

Keywords: **Jumarie's modified R-L fractional derivative, new multiplication, fractional analytic functions, matrix fractional functions, fractional binomial theorem.**

I. INTRODUCTION

In recent decades, the applications of fractional calculus in various fields of science is growing rapidly, such as physics, biology, mechanics, electrical engineering, viscoelasticity, control theory, modelling, economics, etc [1-10]. However, the definition of fractional derivative is not unique. Common definitions include Riemann-Liouville (R-L) fractional derivative, Caputo fractional derivative, Grunwald-Letnikov (G-L) fractional derivative, and Jumarie's modified R-L fractional derivative [11-15]. Since Jumarie type of R-L fractional derivative helps to avoid non-zero fractional derivative of constant function, it is easier to use this definition to connect fractional calculus with classical calculus.

In this paper, based on Jumarie type of R-L fractional derivative and a new multiplication of fractional analytic functions, we can find arbitrary order fractional derivative of the following two matrix fractional functions:

$$E_\alpha\left(Acos_\alpha(x^\alpha)\right),$$

and

$$E_\alpha\left(Asin_\alpha(x^\alpha)\right),$$

where $0 < \alpha \leq 1$, and A is a matrix. Fractional binomial theorem plays an important role in this article. Moreover, our results are generalizations of classical calculus results.

II. PRELIMINARIES

At first, we introduce the fractional derivative used in this paper.

Definition 2.1 ([16])**:** Let $0 < \alpha \leq 1$, and x_0 be a real number. The Jumarie's modified Riemann-Liouville (R-L) α-fractional derivative is defined by

$$\left({}_{x_0}D_x^\alpha\right)[f(x)] = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_{x_0}^{x}\frac{f(t)-f(x_0)}{(x-t)^\alpha}dt\ , \tag{1}$$

where $\Gamma(\ )$ is the gamma function. On the other hand, for any positive integer p, we define $\left({}_{x_0}D_x^{\alpha}\right)^p[f(x)] = \left({}_{x_0}D_x^{\alpha}\right)\left({}_{x_0}D_x^{\alpha}\right)\cdots\left({}_{x_0}D_x^{\alpha}\right)[f(x)]$, the p-th order α-fractional derivative of $f(x)$.

Proposition 2.2 ([17]): *If α, β, x_0, C are real numbers and $\beta \geq \alpha > 0$, then*

$$\left({}_{x_0}D_x^{\alpha}\right)\left[(x-x_0)^{\beta}\right] = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}(x-x_0)^{\beta-\alpha}, \tag{2}$$

and

$$\left({}_{x_0}D_x^{\alpha}\right)[C] = 0. \tag{3}$$

In the following, we introduce the definition of fractional analytic function.

Definition 2.3 ([18]): Let x, x_0, and a_k be real numbers for all k, $x_0 \in (a,b)$, and $0 < \alpha \leq 1$. If the function $f_\alpha: [a,b] \to R$ can be expressed as an α-fractional power series, that is, $f_\alpha(x^\alpha) = \sum_{k=0}^{\infty} \frac{a_k}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha}$ on some open interval containing x_0, then we say that $f_\alpha(x^\alpha)$ is α-fractional analytic at x_0. In addition, if $f_\alpha: [a,b] \to R$ is continuous on closed interval $[a,b]$ and it is α-fractional analytic at every point in open interval (a,b), then f_α is called an α-fractional analytic function on $[a,b]$.

Next, we introduce a new multiplication of fractional analytic functions.

Definition 2.4 ([19]): If $0 < \alpha \leq 1$. Assume that $f_\alpha(x^\alpha)$ and $g_\alpha(x^\alpha)$ are two α-fractional power series at $x = x_0$,

$$f_\alpha(x^\alpha) = \sum_{k=0}^{\infty} \frac{a_k}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha}, \tag{4}$$

$$g_\alpha(x^\alpha) = \sum_{k=0}^{\infty} \frac{b_k}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha}. \tag{5}$$

Then

$$\begin{aligned} & f_\alpha(x^\alpha) \otimes_\alpha g_\alpha(x^\alpha) \\ & = \sum_{k=0}^{\infty} \frac{a_k}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha} \otimes_\alpha \sum_{k=0}^{\infty} \frac{b_k}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha} \\ & = \sum_{k=0}^{\infty} \frac{1}{\Gamma(k\alpha+1)}\left(\sum_{m=0}^{k}\binom{k}{m} a_{k-m} b_m\right)(x-x_0)^{k\alpha}. \end{aligned} \tag{6}$$

Equivalently,

$$\begin{aligned} & f_\alpha(x^\alpha) \otimes_\alpha g_\alpha(x^\alpha) \\ & = \sum_{k=0}^{\infty} \frac{a_k}{k!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes_\alpha k} \otimes_\alpha \sum_{k=0}^{\infty} \frac{b_k}{k!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes_\alpha k} \\ & = \sum_{k=0}^{\infty} \frac{1}{k!}\left(\sum_{m=0}^{k}\binom{k}{m} a_{k-m} b_m\right)\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes_\alpha k}. \end{aligned} \tag{7}$$

Definition 2.5 ([20]): If $0 < \alpha \leq 1$, and A is a matrix. The matrix α-fractional exponential function is defined by

$$E_\alpha(Ax^\alpha) = \sum_{k=0}^{\infty} A^k \frac{x^{k\alpha}}{\Gamma(k\alpha+1)} = \sum_{k=0}^{\infty} \frac{1}{k!}\left(A\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha k}. \tag{8}$$

In addition, the matrix α-fractional cosine and matrix α-fractional sine function are defined as follows:

$$cos_\alpha(Ax^\alpha) = \sum_{k=0}^{\infty} A^{2k} \frac{(-1)^k x^{2k\alpha}}{\Gamma(2k\alpha+1)} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!}\left(A\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha 2k}, \tag{9}$$

and

$$sin_\alpha(Ax^\alpha) = \sum_{k=0}^{\infty} A^{2k+1} \frac{(-1)^k x^{(2k+1)\alpha}}{\Gamma((2k+1)\alpha+1)} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!}\left(A\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha (2k+1)}. \tag{10}$$

Theorem 2.6 (fractional Euler's formula): *If $0 < \alpha \leq 1$, and $i = \sqrt{-1}$, then*

$$E_\alpha(ix^\alpha) = cos_\alpha(x^\alpha) + isin_\alpha(x^\alpha). \tag{11}$$

Theorem 2.7 (fractional DeMoivre's formula)**:** *If* $0 < \alpha \leq 1$, *and* k *is an integer, then*

$$[cos_\alpha(x^\alpha) + isin_\alpha(x^\alpha)]^{\otimes_\alpha k} = cos_\alpha(kx^\alpha) + isin_\alpha(kx^\alpha). \quad (12)$$

Definition 2.8: The smallest positive real number T_α such that $E_\alpha(iT_\alpha) = 1$, is called the period of $E_\alpha(ix^\alpha)$.

Theorem 2.9 (fractional binomial theorem)**:** *If* $0 < \alpha \leq 1$, k *is a nonnegative integer, and* $f_\alpha(x^\alpha)$, $g_\alpha(x^\alpha)$ *are two* α-*fractional analytic functions. Then*

$$[f_\alpha(x^\alpha) + g_\alpha(x^\alpha)]^{\otimes_\alpha k} = \sum_{m=0}^{k} \binom{k}{m} \left(f_\alpha(x^\alpha)\right)^{\otimes_\alpha (k-m)} \otimes_\alpha \left(g_\alpha(x^\alpha)\right)^{\otimes_\alpha m}, \quad (13)$$

where $\binom{m}{k} = \frac{m!}{k!(m-k)!}$.

III. MAIN RESULTS

In this section, we obtain arbitrary order fractional derivative of two matrix fractional functions. At first, we need a lemma.

Lemma 3.1: *If* $0 < \alpha \leq 1$, *and* A *is a matrix, then*

$$E_\alpha\left(Acos_\alpha(x^\alpha)\right) = \sum_{k=0}^{\infty} \frac{1}{k!2^k} A^k \sum_{m=0}^{k} \binom{k}{m} cos_\alpha((k-2m)x^\alpha), \quad (14)$$

and

$$E_\alpha\left(Asin_\alpha(x^\alpha)\right) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!2^k} A^k \sum_{m=0}^{k} \binom{k}{m} \left[cos\frac{k\pi}{2} \cdot cos_\alpha((k-2m)x^\alpha) - sin\frac{k\pi}{2} \cdot sin_\alpha((k-2m)x^\alpha)\right]. \quad (15)$$

Proof: $E_\alpha\left(Acos_\alpha(x^\alpha)\right)$

$= \sum_{k=0}^{\infty} \frac{1}{k!} \left(Acos_\alpha(x^\alpha)\right)^{\otimes_\alpha k}$

$= \sum_{k=0}^{\infty} \frac{1}{k!} A^k \left(\frac{1}{2}[E_\alpha(ix^\alpha) + E_\alpha(-ix^\alpha)]\right)^{\otimes_\alpha k}$ (by fractional Euler's formula)

$= \sum_{k=0}^{\infty} \frac{1}{k!2^k} A^k \sum_{m=0}^{k} \binom{k}{m} \left(E_\alpha(ix^\alpha)\right)^{\otimes_\alpha (k-m)} \otimes_\alpha \left(E_\alpha(-ix^\alpha)\right)^{\otimes_\alpha m}$ (by fractional binomial theorem)

$= \sum_{k=0}^{\infty} \frac{1}{k!2^k} A^k \sum_{m=0}^{k} \binom{k}{m} E_\alpha(i(k-m)x^\alpha) \otimes_\alpha E_\alpha(-imx^\alpha)$ (by fractional DeMoivre's formula)

$= \sum_{k=0}^{\infty} \frac{1}{k!2^k} A^k \sum_{m=0}^{k} \binom{k}{m} E_\alpha(i(k-2m)x^\alpha)$

$= \sum_{k=0}^{\infty} \frac{1}{k!2^k} A^k \sum_{m=0}^{k} \binom{k}{m} cos_\alpha((k-2m)x^\alpha)$.

On the other hand,

$E_\alpha\left(Asin_\alpha(x^\alpha)\right)$

$= \sum_{k=0}^{\infty} \frac{1}{k!} \left(Asin_\alpha(x^\alpha)\right)^{\otimes_\alpha k}$

$= \sum_{k=0}^{\infty} \frac{1}{k!} A^k \left(\frac{1}{2i}[E_\alpha(ix^\alpha) - E_\alpha(-ix^\alpha)]\right)^{\otimes_\alpha k}$ (by fractional Euler's formula)

$= \sum_{k=0}^{\infty} \frac{1}{k!2^k} (-i)^k A^k \sum_{m=0}^{k} \binom{k}{m} \left(E_\alpha(ix^\alpha)\right)^{\otimes_\alpha (k-m)} \otimes_\alpha \left(E_\alpha(-ix^\alpha)\right)^{\otimes_\alpha m}$ (by fractional binomial theorem)

$= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!2^k} \left(cos\frac{k\pi}{2} + isin\frac{k\pi}{2}\right) A^k \sum_{m=0}^{k} \binom{k}{m} E_\alpha(i(k-m)x^\alpha) \otimes_\alpha E_\alpha(-imx^\alpha)$ (by fractional DeMoivre's formula)

$= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!2^k} \left(cos\frac{k\pi}{2} + isin\frac{k\pi}{2}\right) A^k \sum_{m=0}^{k} \binom{k}{m} E_\alpha(i(k-2m)x^\alpha)$

$= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!2^k} \left(cos\frac{k\pi}{2} + isin\frac{k\pi}{2}\right) A^k \sum_{m=0}^{k} \binom{k}{m} [cos_\alpha((k-2m)x^\alpha) + isin_\alpha((k-2m)x^\alpha)]$ (by fractional Euler's formula)

$= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!2^k} A^k \sum_{m=0}^{k} \binom{k}{m} \left[cos\frac{k\pi}{2} \cdot cos_\alpha((k-2m)x^\alpha) - sin\frac{k\pi}{2} \cdot sin_\alpha((k-2m)x^\alpha)\right]$. q.e.d.

Theorem 3.2: *Suppose that* $0 < \alpha \leq 1$, *p is any positive integer, and A is a matrix, then*

$$\left({}_0D_x^\alpha\right)^p [E_\alpha(Acos_\alpha(x^\alpha))] = \sum_{k=0}^{\infty} \frac{1}{k!2^k} A^k \sum_{m=0}^{k} \binom{k}{m} (k-2m)^p cos_\alpha\left((k-2m)x^\alpha + p \cdot \frac{T_\alpha}{4}\right) . \quad (16)$$

and

$$\left({}_0D_x^\alpha\right)^p [E_\alpha(Asin_\alpha(x^\alpha))]$$
$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!2^k} A^k \sum_{m=0}^{k} \binom{k}{m} (k-2m)^p \left[cos\frac{k\pi}{2} \cdot cos_\alpha\left((k-2m)x^\alpha + p \cdot \frac{T_\alpha}{4}\right) - sin\frac{k\pi}{2} \cdot sin_\alpha\left((k-2m)x^\alpha + p \cdot \frac{T_\alpha}{4}\right)\right] . \quad (17)$$

Proof By Lemma 3.1,

$$\left({}_0D_x^\alpha\right)^p [E_\alpha(Acos_\alpha(x^\alpha))]$$
$$= \left({}_0D_x^\alpha\right)^p \left[\sum_{k=0}^{\infty} \frac{1}{k!2^k} A^k \sum_{m=0}^{k} \binom{k}{m} cos_\alpha((k-2m)x^\alpha)\right]$$
$$= \sum_{k=0}^{\infty} \frac{1}{k!2^k} A^k \sum_{m=0}^{k} \binom{k}{m} \left({}_0D_x^\alpha\right)^p [cos_\alpha((k-2m)x^\alpha)]$$
$$= \sum_{k=0}^{\infty} \frac{1}{k!2^k} A^k \sum_{m=0}^{k} \binom{k}{m} (k-2m)^p cos_\alpha\left((k-2m)x^\alpha + p \cdot \frac{T_\alpha}{4}\right) .$$

And

$$\left({}_0D_x^\alpha\right)^p [E_\alpha(Asin_\alpha(x^\alpha))]$$
$$= \left({}_0D_x^\alpha\right)^p \left[\sum_{k=0}^{\infty} \frac{(-1)^k}{k!2^k} A^k \sum_{m=0}^{k} \binom{k}{m} \left[cos\frac{k\pi}{2} \cdot cos_\alpha((k-2m)x^\alpha) - sin\frac{k\pi}{2} \cdot sin_\alpha((k-2m)x^\alpha)\right]\right]$$
$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!2^k} A^k \sum_{m=0}^{k} \binom{k}{m} \left({}_0D_x^\alpha\right)^p \left[cos\frac{k\pi}{2} \cdot cos_\alpha((k-2m)x^\alpha) - sin\frac{k\pi}{2} \cdot sin_\alpha((k-2m)x^\alpha)\right]$$
$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!2^k} A^k \sum_{m=0}^{k} \binom{k}{m} (k-2m)^p \left[cos\frac{k\pi}{2} \cdot cos_\alpha\left((k-2m)x^\alpha + p \cdot \frac{T_\alpha}{4}\right) - sin\frac{k\pi}{2} \cdot sin_\alpha\left((k-2m)x^\alpha + p \cdot \frac{T_\alpha}{4}\right)\right] .$$

q.e.d.

IV. CONCLUSION

In this paper, based on Jumarie's modified R-L fractional derivative and a new multiplication of fractional analytic functions, we can obtain arbitrary order fractional derivative of two matrix fractional functions. Fractional binomial theorem plays an important role in this article. In fact, our results are generalizations of classical calculus results. In the future, we will continue to use our methods to study the problems in fractional differential equations and applied mathematics.

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